From simple to complicated

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Published in:
Linear Algebra and Its Applications

DOI:
10.1016/0024-3795(90)90314-3

Publication date:
1990

From Simple to Complicated: Noncausal Bounded Input, Bounded Output Stability in Linear Discrete Time Models in the Deterministic and Stochastic Cases

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ABSTRACT

We consider the question of bounded input, bounded output stability for time-invariant linear systems with a finite dimensional state space and with time axis $\mathbb{Z}$, i.e., all the integers. Our approach also includes the stochastic ARMA models. We do not assume that the inputs are necessarily nonanticipating, and in this respect our results differ from most existing ones. Similar results have been given by Hannan and Deistler and by Brockwell and Davis. Our approach is polynomial-algebra-oriented, and does not use strictly rational functions.

NOTATION

$\mathbb{Z} :=$ {integers}; $\mathbb{R} :=$ {reals}; $\mathbb{C} :=$ {complex numbers}. With $q \in \mathbb{Z}$, $q \geqslant 0$, $\mathbb{R}^q$ is the q-dimensional Euclidean space. $(\mathbb{R}^q)^{\mathbb{Z}} := \{\omega : \mathbb{Z} \to \mathbb{R}^q\}$, i.e. the set of time sequences with elements in $\mathbb{R}^q$. By $\sigma : (\mathbb{R}^q)^{\mathbb{Z}} \to (\mathbb{R}^q)^{\mathbb{Z}}$ we denote the shift operator $(\sigma w)(t) := \omega(t+1)\ \forall t \in \mathbb{Z}$. The operator σ^0 is assumed to be the identity, and for all $k \in \mathbb{Z}$ we define the operator σ^k by $\sigma^k := \sigma \cdot \sigma^{k-1}$. $\mathbb{R}[s] :=$ {polynomials with indeterminate s and coefficients in $\mathbb{R}$}. $\mathbb{R}^{k \times l}[s] :=$ {matrices with k rows and l columns and with all elements in $\mathbb{R}[s]$}.

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INTRODUCTION

The following object is well known in linear systems theory. Let A, B, C, and D be matrices with a finite number of rows and columns and with all elements in $\mathbb{R}$. We define the object $\mathfrak{B}$ as follows:

$$\mathfrak{B} := \Big\{(y,u) \in (\mathbb{R}^p \times \mathbb{R}^m)^{\mathbb{Z}} \Big| \exists x \in (\mathbb{R}^n)^{\mathbb{Z}}$$

$$\text{such that } \sigma x = Ax + Bu,\ y = Cx + Du\Big\}.$$

Let us define $q := p + m$ and $\omega := (y,u)$. Then $\mathfrak{B}$ is a set of trajectories $\omega : \mathbb{Z} \to \mathbb{R}^q$ with the following properties:

(1) $\mathfrak{B}$ is a linear subspace of $(\mathbb{R}^q)^{\mathbb{Z}}$.
(2) $\sigma\mathfrak{B} = \mathfrak{B}$, i.e. $\mathfrak{B}$ is shift-invariant.
(3) $\mathfrak{B}$ is closed in the topology of pointwise convergence.

Actually one can also prove the following (see [3]).

THEOREM. *The following statements are equivalent*:

(i) $\mathfrak{B} \subseteq (\mathbb{R}^q)^{\mathbb{Z}}$ *is linear, shift-invariant, and closed.*

(ii) *There is a nonnegative number g and a matrix* $R(s) \in \mathbb{R}^{g \times q}[s]$ *such that*

$$\mathfrak{B} = \{\omega \,|\, R(\sigma)\omega = 0\}.$$

(iii) *There is a partition of the variables,* $\omega = (y,u)$, *and there is a nonnegative integer n, and there are real matrices A, B, C, and D such that* $\mathfrak{B} = \{(y,u) | \exists x \in (\mathbb{R}^n)^{\mathbb{Z}}$ *with* $\sigma x = Ax + Bu,\ y = Cx + Du\}$.

(iv) *Given the partition in* (iii), *there are polynomial matrices* $P(s)$ *and* $Q(s)$ *such that* $[P(s)]^{-1}Q(s)$, *a matrix with rational elements, has only proper elements. That is, let* $e(s) := p(s)/q(s)$, *with* $p(s)$ *and* $q(s)$ *polynomials, be an element of* $P(s)^{-1}Q(s)$; *then the degree of* $p(s)$ *is at most equal to the degree of* $q(s)$. *Further,* $\mathfrak{B} = \{(y,u) | P(\sigma)y = Q(\sigma)u\}$.

(v) *Given a representation of* $\mathfrak{B}$ *as in* (iv), *with* $\omega = (y,u)$, *one can find a nonnegative integer n and matrices A, B, C, and D such that* $\mathfrak{B} = \{(y,u) | \exists x \in (\mathbb{R}^n)^{\mathbb{Z}}$ *with* $\sigma x = Ax + Bu,\ y = Cx + Du\}$.

In (iii) we have the familiar input state output representation, whereas (iv) is partly stated in terms of a *transfer matrix* $P(s)^{-1}Q(s)$. We would like to stress in this connection that knowledge of a transfer matrix does not necessarily completely specify the set $\mathfrak{B}$. To illustrate this we consider in $(\mathbb{R}^2)^{\mathbb{Z}}$ the following two behaviors $\mathfrak{B}_1$ and $\mathfrak{B}_2$.

$$\mathfrak{B}_1 := \left\{(y,u) \in (\mathbb{R}^2)^{\mathbb{Z}} \middle| y = u\right\},$$

$$\mathfrak{B}_2 := \left\{(y,u) \in (\mathbb{R}^2)^{\mathbb{Z}} \middle| (\sigma - 2)y = (\sigma - 2)u\right\}.$$

It is evident that $\mathfrak{B}_1 \neq \mathfrak{B}_2$ but their transfer functions are equal to 1.

In the sequel we will study the following stability problem. Let $\mathfrak{B} \subseteq (\mathbb{R}^q)^{\mathbb{Z}}$ be given by

$$\mathfrak{B} = \{(y,u) | P(\sigma)y = Q(\sigma)u\},$$

where it is only assumed that det $P(s) \neq 0$, so we do not necessarily have that $P(s)^{-1}Q(s)$ is proper.

Problem 1. Give, if possible, necessary and sufficient conditions on $(P(s), Q(s))$ such that for every bounded sequence u [i.e., $\exists k > 0$ such that $|u(t)| \leqslant k \ \forall t \in \mathbb{Z}$] there is a bounded sequence y such that $(y,u) \in \mathfrak{B}$.

Two remarks are in order here:

(1) A preliminary question is of course whether there is for every sequence u a sequence y such that $(y,u) \in \mathfrak{B}$. The answer is yes in case $P(s)^{-1}$ exists as a rational matrix. We will come to this question later on.

(2) Our problem differs from the classical stability problems in the following sense. In the first place, we do not assume that $P(s)^{-1}Q(s)$ is proper, i.e., that u is a *causal* input. Secondly, even when $P(s)^{-1}Q(s)$ is proper, we do not demand that u be nonanticipating with respect to y. So the concept of causality is a completely absent in our problem statement.

In solving this problem we discovered that with a little bit of extra work we also could elucidate the following, *stochastic* problem.

Problem 2. Let $P(s)$ be a square matrix with nonzero determinant. $Q(s)$ is also a polynomial matrix. Give, if possible, necessary and sufficient conditions such that for every bounded stochastic process u there is a bounded stochastic process y such that

$$P(\sigma)y = Q(\sigma)u.$$

To be precise, for all $t \in \mathbb{Z}$ let $u(t)$ be a stochastic variable defined on a given probability space $(\Omega, \mathscr{P})$, where $\mathscr{P}$ is a probability measure on Ω with values in $\mathbb{R}^m$. We assume that u is *bounded* in the sense that for some nonnegative $k \in \mathbb{R}$

$$E\|z(t)\|^2 \leqslant k \qquad \forall t \in \mathbb{Z},$$

where E denotes the mathematical expectation. We are asking for conditions on P and Q such that there is a stochastic process y defined on the same probability space such that $E\|y(t)\|^2 \leqslant \tilde{k}$ $\forall t \in \mathbb{Z}$, for some $\tilde{k} \in \mathbb{R}$ and with $P(\sigma)y = Q(\sigma)u$. The equations $P(\sigma)y = Q(\sigma)u$ constitute a stochastic ARMA model.

Again, the notion of causality is absent, and so this problem statement is also new.

Both problems will be attacked in a *polynomial* way, meaning that our most important tools are from polynomial algebra. In this respect the most difficult theorem we use is about the Smith form of a polynomial matrix (see [4]). Our way of approaching these problems is in line with [3].

This paper is called "From Simple to Complicated, etc." because we start with a treatment of both problems in the case wherein $P(s)$ and $Q(s)$ are polynomials, i.e., elements of $\mathbb{R}[s]$. The results of the scalar case will be used to attack the general problems.

SCALAR CASE, STOCHASTIC PART

We now consider the case $p(\sigma)y = q(\sigma)u$, where $p(s)$ and $q(s)$ are assumed to be polynomials such that $p(s) \neq 0$ and such that $p(s), q(s)$ are coprime.

THEOREM. *Let $u : \mathbb{Z} \to \mathbb{R}$ be a given stochastic process with the following two properties:*

(a) *There are numbers $k_1, k_2 \in \mathbb{R}$, both greater than zero, such that for all $t \in \mathbb{Z}$ the following holds:*

$$k_1 \leqslant E\|u(t)\|^2 \leqslant k_2.$$

(b) $\forall t, t' \in \mathbb{Z}$, $t \neq t'$ *we have $E(u(t)u(t')) = 0$, i.e., the random variables $\{u(t)\}$ are uncorrelated.*

Then the following holds: There is a bounded stochastic process y with $p(\sigma)y = q(\sigma)u$ precisely when $p(\lambda) = 0$, $\lambda \in \mathbb{C}$, implies $|\lambda| \neq 1$. Whatever $p(s)$ is, there is at most one bounded process y with $p(\sigma)y = q(\sigma)u$.

Proof. First we assume that $\lambda \in \mathbb{C}$ has modulus 1 and that $p(\lambda) = 0$. Suppose to the contrary that there is a bounded stochastic process y with

$$p(\sigma)y = q(\sigma)u.$$

Write $p(s) = (s - \lambda)\tilde{p}(s)$, and define $\tilde{y} := \tilde{p}(\sigma)y$. Then

$$(\sigma - \lambda)\tilde{y} = q(\sigma)u$$

and $\tilde{y}$ (now possibly complex valued) is still bounded. As $p(s)$ and $q(s)$ are assumed to be coprime, there is a number $c_0 \in \mathbb{C}$ with $c_0 \neq 0$ and a polynomial $\gamma(s) \in \mathbb{C}[s]$ with $q(s) = \gamma(s)(s - \lambda) + c_0$. Hence

$$(\sigma - \lambda)\tilde{y} = [\gamma(\sigma)(\sigma - \lambda) + c_0]u.$$

Notice that by assumption $\tilde{\tilde{y}} := \tilde{y} - \gamma(\sigma)u$ is also bounded; hence

$$(\sigma - \lambda)\tilde{\tilde{y}} = c_0 u =: \tilde{\tilde{u}}.$$

Note that $\tilde{\tilde{u}}$ is also uncorrelated and at the same time bounded from above and bounded away from zero.

In the rest of the first half of the proof, however, we will show that $(\sigma - \lambda)v = \tilde{\tilde{u}}$ cannot have a bounded solution v. Remark that $(\sigma - \lambda)v = 0$ implies that v is a bounded stochastic process. Hence it suffices to construct a stochastic process v such that $(\sigma - \lambda)v = \tilde{\tilde{u}}$ and such that v is not bounded. Define $v(0) := \tilde{\tilde{u}}(0)$, $v(t+1) := \lambda v(t) + \tilde{\tilde{u}}(t)$ $\forall t \geqslant 0$, and $v(t) := \lambda^{-1}[v(t+1) - \tilde{\tilde{u}}(t)]$ $\forall t \leqslant -1$. Then trivially $(\sigma - \lambda)v = \tilde{\tilde{u}}$. For all $t \geqslant 1$ we have

$$v(t) = (\lambda^t + \lambda^{t-1})u(0) + \lambda^{t-2}u(1) + \lambda^{t-3}u(2) + \cdots + u(t-1),$$

and as the stochastic process u is bounded away from zero *and* uncorrelated, it follows that the stochastic process v constructed this way is unbounded.

Now we assume that $p(s) = \prod_{i=1}^{n}(s - \lambda_i)$ with $\lambda_i \in \mathbb{C}$ such that $|\lambda_i| \neq 1$ $\forall i \in \{1,2,\ldots,n\}$. By induction we will prove that there is precisely one bounded stochastic process y with $p(\sigma)y = q(\sigma)u$. First we consider the case $n = 1$ and distinguish $|\lambda_1| < 1$ from $|\lambda_1 > 1$. In the first case we define, $\forall t \in \mathbb{Z}$,

$$y(t) := \sum_{i=0}^{\infty} \lambda_1^i u(t-i-1).$$

In the second case we define, $\forall t \in \mathbb{Z}$,

$$y(t) := -\sum_{i=1}^{\infty} \lambda_1^{-i} u(t+i-1).$$

In both cases we have by construction that $(\sigma - \lambda_1)y = u$ and that y is bounded from above.

We now consider the general case $(\sigma-\lambda_1)(\sigma-\lambda_2)\cdots(\sigma-\lambda_n)y = u$, where u is given. First we construct a bounded process y_1 such that $(\sigma-\lambda_1)y_1 = u$. Then we construct a bounded process y_2 such that $(\sigma-\lambda_2)y_2 = y_1$. Continuing this way, we finally construct a bounded process y_n such that $(\sigma-\lambda_n)y_n = y_{n-1}$. Now it is easy to show that $(\sigma-\lambda_1)(\sigma-\lambda_2)\cdots(\sigma-\lambda_n)y_n = u$. As in the case under discussion $p(\sigma)y = 0$ can have *no* bounded solution different from zero, it follows that $p(\sigma)y = u$ has precisely one bounded solution y, and we are done with the proof. ■

An example that bears a close resemblance to this result is treated on pp. 79–81 of [2].

SCALAR CASE, DETERMINISTIC PART

Again we take coprime polynomials $p(s)$ and $q(s)$ such that $p(s) \neq 0$.

THEOREM. *For all bounded sequences $u: \mathbb{Z} \to \mathbb{R}$ there is a bounded sequence y such that $p(\sigma)y = q(\sigma)u$ precisely when $p(\lambda) = 0$, and $\lambda \in \mathbb{C}$ implies $|\lambda| \neq 1$. When $p(\lambda) = 0$ has no roots on the unit circle, then the equation $p(\sigma)y = q(\sigma)u$ has precisely one bounded solution y for every bounded sequence u.*

Proof. Let us assume that $p(s) = \prod_{i=1}^{n}(s-\lambda_i)$, where $|\lambda_i| \neq 1$ $\forall i \in \{1, 2, \ldots, n\}$. In this case we can repeat almost verbatim the corresponding part of the foregoing theorem. So there remains to consider the case wherein there is a $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ and with $p(\lambda) = 0$. We will now construct a *bounded* sequence $\hat{u}$ and an *unbounded* sequence y such that $(\sigma - \lambda)y = \hat{u}$. As $(\sigma - \lambda)y = 0$ implies that y is bounded, this immediately implies that $p(\sigma)y = \hat{u}$ has no bounded solution.

Now we turn to the promised construction. Take $\hat{u}: \mathbb{Z} \to \mathbb{R}$ such that $\hat{u}(t) = 0$ $\forall t \leqslant -1$. Let $y: \mathbb{Z} \to \mathbb{C}$ be such that $y(t) = 0$ $\forall t \leqslant 0$. We define $\hat{u}(0) := 1$ and $y(1) := 1$. The other elements of $y: \mathbb{Z} \to \mathbb{C}$ and $\hat{u}: \mathbb{Z} \to \mathbb{R}$ are recursively defined as follows: Take $\hat{u}(1)$ to be equal to -1 or $+1$, but such that the angle between $\lambda y(1)$ and $\hat{u}(1)$ is nonobtuse. As $y(2) = \lambda y(1) + \hat{u}(1)$, this implies that $|y(2)| \geqslant \sqrt{2}$. Assume that $\hat{u}(1)$, $\hat{u}(2), \dots, \hat{u}(t-2)$ and $y(1), y(2), \dots, y(t-1)$ are chosen such that $|y(t-1)| \geqslant \sqrt{t-1}$, $t \geqslant 2$. Take now $\hat{u}(t-1)$ such that the angle between $\lambda y(t-1)$ and $\hat{u}(t-1)$ is nonobtuse and such that $\hat{u}(t-1) = +1$ or -1. Define $y(t) := \lambda y(t-1) + \hat{u}(t-1)$; then we have $|y(t)| \geqslant \sqrt{t}$, and we are done with the proof. ∎

MORE ABOUT GENERAL ARMA MODELS

Before continuing with our stability problems we remark the following. Let $\mathfrak{B} \subseteq (\mathbb{R}^q)^{\mathbb{Z}}$ be linear, shift-invariant, and closed. Then $\mathfrak{B}$ can be written as the kernel of a polynomial operator. It is easy to see that without loss of generality we may take $R(s) \in \mathbb{R}^{g \times q}[s]$ such that the row rank of $R(s)$ (over the rational functions) is equal to g and where $\mathfrak{B} = \{\omega \mid R(\sigma)\omega = 0\}$. When we assume in addition that rank $R(\lambda) = g$ $\forall \lambda \in \mathbb{C}$, $\lambda \neq 0$, then $\mathfrak{B}$ is said to be *controllable* (see [3]). One can prove that this notion is equivalent to the classical notion of controllability. When instead $R(s)$ is a square matrix such that $R(s)^{-1}$ exists as a rational matrix, then $\mathfrak{B}$ is said to be *autonomous* (see [3]). One can prove (see [4]) that every linear, closed, and shift-invariant $\mathfrak{B} \subseteq (\mathbb{R}^q)^{\mathbb{Z}}$ can be written as the sum of a controllable $\mathfrak{B}_1$ and an autonomous $\mathfrak{B}_2$, where $\mathfrak{B}_i$ is closed, linear, and shift-invariant, $i = 1, 2$. One can also prove (see [5]) that this $\mathfrak{B}_1$ is unique for a given $\mathfrak{B}$, and hence we call $\mathfrak{B}_1$ the controllable part of $\mathfrak{B}$.

Assume that $\mathfrak{B} = \{\omega \mid R(\sigma)\omega = 0\}$, where rank $R(\lambda) = g$ $\forall 0 \neq \lambda \in \mathbb{C}$, and where $R(s \in \mathbb{R}^{g \times q}[s]$. It is well known (see for instance [6]) that this implies the existence of a matrix $A(s) \in \mathbb{R}^{(q-g) \times q}[s]$ such that the matrix

$$U(s) := \begin{bmatrix} R(s) \\ A(s) \end{bmatrix}$$

is unimodular, i.e., $\det U(s) = c_0 s^k$ for some $c_0 \neq 0$ and some $k \in \mathbb{Z}$. This immediately implies the existence of a polynomial matrix $M(s)$ such that $\mathfrak{B} = \{\omega \mid \exists a \text{ with } \omega = M(\sigma)a\}$. This is a so-called *moving average* representation of $\mathfrak{B}$, whereas the kernel representation is called *autoregressive*. Hence a linear time-invariant finite dimensional discrete time system is controllable precisely when it admits a moving average representation. Suppose now that $\mathfrak{B} = \{(y,u) \mid P(\sigma)y = Q(\sigma)u\}$, where $P(s)$ is square and has a nonzero determinant. Then the following holds: For all sequences u there is a sequence y such that $P(\sigma)y = Q(\sigma)u$. This follows from these two observations:

(1) There are unimodular matrices $U(s)$ and $V(s)$ such that $U(s)P(s)V(s)$ is diagonal (Smith form; see [4]).

(2) Let $r(s) \in \mathbb{R}[s]$ be nonzero. Then for every sequence ζ there is a sequence y with $r(\sigma)y = \zeta$ [2].

The equation $P(\sigma)y = Q(\sigma)u$ is called *autoregressive moving average* (ARMA for short).

For more relations of these polynomial equations to linear systems theory the reader is referred to [3] and [5].

STABILITY IN THE GENERAL CASE, DETERMINISTIC PART

First we consider the equation $P(\sigma)y = Q(\sigma)u$, and we assume that $P(s)$ and $Q(s)$ are coprime and that $\det P(s) \neq 0$.

THEOREM. *For all bounded sequences u, there is a bounded sequence y with $P(\sigma)y = Q(\sigma)u$ precisely when* $\det P(\lambda) = 0$, $\lambda \in \mathbb{C}$, *implies* $|\lambda| \neq 1$. *In this case there is for every bounded u precisely one bounded y with $P(\sigma)y = Q(\sigma)u$.*

Proof. First we assume that $\det P(\lambda) = 0$ implies $|\lambda| \neq 1$. Take unimodular matrices $U(s)$ and $V(s)$ such that $\Delta(s) := U(s)P(s)V(s)$ is diagonal. $P(\sigma)y = Q(\sigma)u$ is easily seen to be equivalent to $\Delta(\sigma)V^{-1}(\sigma)y = U(\sigma)Q(\sigma)u$. Applying the result in the scalar case (recall that Δ is diagonal) immediately leads to the result that for all bounded u there is precisely one bounded y with $P(\sigma)y = Q(\sigma)u$.

We now assume that there is a $\lambda \in \mathbb{C}$ with $|\lambda| = 1$, and $\det P(\lambda) = 0$. Now assume to the contrary that for all bounded sequences u there is a bounded sequence y such that $P(\sigma)y = Q(\sigma)u$. As $P(s)$ and $Q(s)$ are coprime, there are polynomial matrices $A(s)$ and $B(s)$ such that

$$U(s) := \begin{pmatrix} P(s) & -Q(s) \\ A(s) & B(s) \end{pmatrix}$$

is unimodular, and hence we may write, for some $k \in \mathbb{Z}$,

$$s^k U(s)^{-1} = \begin{pmatrix} T(s) & \begin{matrix} M(s) \\ N(s) \end{matrix} \end{pmatrix}$$

for suitable polynomial matrices $T(s)$, $M(s)$, and $N(s)$. One further can prove that without loss of generality one can take $N(s)$ such that $\det N(s) = \det P(s)$.

Let y and u be such that $P(\sigma)y = Q(\sigma)u$, and both y and u be bounded. Then of course $\zeta := A(\sigma)y + B(\sigma)u$ is also bounded. Hence we have

$$U(\sigma)\begin{pmatrix} y \\ u \end{pmatrix} = \begin{pmatrix} 0 \\ \zeta \end{pmatrix}$$

and therefore

$$\begin{pmatrix} y \\ u \end{pmatrix} = \begin{pmatrix} T(\sigma) & \begin{matrix} M(\sigma) \\ N(\sigma) \end{matrix} \end{pmatrix} \begin{pmatrix} 0 \\ \zeta \end{pmatrix} = \begin{pmatrix} M(\sigma)\zeta \\ N(\sigma)\zeta \end{pmatrix}.$$

This implies that the assumption we made in the second half of the proof is equivalent to the following statement: For all bounded sequences u there exists a bounded sequence ζ such that $u = N(\sigma)\zeta$. By applying the Smith form theorem to $N(\sigma)$ and invoking the results in the scalar case we arrive at a contradiction. For suppose that for some unimodular matrices $U(s)$ and $V(s)$ we have that $U(s)N(s)V(s)$ is a diagonal matrix, say $\Delta(s)$; then the above statement is equivalent to the following one: For all bounded sequences u there exists a bounded sequence ζ such that

$$u = \Delta(\sigma)\zeta.$$

As $\Delta(s)$ is diagonal, and as at least one of the diagonal elements of $\Delta(s)$ has a root with modulus equal to 1, the result follows from applying the scalar case theorem. ■

We would like to remark that this line of argument has proved even more: When $P(s)$ and $Q(s)$ are coprime and when there is a $\lambda \in \mathbb{C}$ such that $|\lambda| = 1$ and $\det P(\lambda) = 0$, then there is a bounded sequence u and a sequence y with $P(\sigma)y = Q(\sigma)u$ where y is unbounded but such that $y(t)$ increases not faster than the order of t. We need only recall the construction in the scalar part of the story.

Precisely this observation solves the general problem where $\det P(s) \neq 0$ but where $P(s)$ and $Q(s)$ are not necessarily coprime.

COROLLARY (General result). *Let* $\det P(s) \neq 0$. *Let* $P(s) = R(s)\tilde{P}(s)$ *and* $Q(s) = R(s)\tilde{Q}(s)$, *where* $\tilde{P}(s)$ *and* $\tilde{Q}(s)$ *are coprime. Then for all bounded* u *there is a bounded* y *such that* $P(\sigma)y = Q(\sigma)u$ *precisely when* $\det \tilde{P}(\lambda) = 0$ *implies that* $|\lambda| \neq 1$.

Proof. Assume that there is a $\lambda \in \mathbb{C}$ such that $\tilde{P}(\lambda) = 0$ and $|\lambda| = 1$. Then, applying the previous result, it follows that there is a bounded sequence $\hat{u}$ and an unbounded sequence $\hat{y}$, increasing not faster than t, such that $\hat{P}(\sigma)\hat{y} = \hat{Q}(\sigma)\hat{u}$, and hence also $P(\sigma)\hat{y} = Q(\sigma)\hat{u}$. Assume to the contrary that there is a bounded sequence $\tilde{y}$ such that $P(\sigma)\tilde{y} = Q(\sigma)\hat{u}$. Then, of course, $P(\sigma)(\hat{y} - \tilde{y}) = 0$. It is not hard to see that, $\hat{y} - \tilde{y}$ being unbounded, $(\hat{y} - \tilde{y})(t)$ is an exponential function of t. One can prove this by invoking again the theorem about the Smith form or by using a state space realization of the behavior $\{y \mid P(\sigma)y = 0\}$. It is clear that this leads to an contradiction: an exponentially increasing sequence cannot be the sum of a bounded sequence and a sequence increasing not faster than the order of t.

As the other part of the proof is trivial, we omit it, and we are done with the proof. ■

We would like to remark that the corollary says that it is precisely the *controllable* part of the behavior $\{(y, u) \mid P(\sigma)y = Q(\sigma)u\}$ that determines whether the behavior is bounded u, bounded y stable.

STABILITY IN THE GENERAL CASE, STOCHASTIC PART

Repeating the foregoing almost verbatim, one has in the stochastic case the following result.

THEOREM. *Let* $P(s) = R(s)\tilde{P}(s)$ *and* $Q(s) = R(s)\tilde{Q}(s)$, *where* $\det P(s) \neq 0$ *and where* $\tilde{P}(s)$ *and* $\tilde{Q}(s)$ *are coprime. Assume that* $\det \tilde{P}(\lambda) = 0$

implies $|\lambda| \neq 1$. *Then for every bounded stochastic process* u *there is a bounded stochastic process* y *with* $P(\sigma)y = Q(\sigma)u$.

The situation is not as nice as in the scalar case. In order to show that we take the following example:

$$(\sigma - 1)y = u_1 - u_2,$$

where y, u_1 and u_2 are all real valued stochastic processes. When we take $u_1 = u_2$, then there is still a bounded sequence y with $(\sigma - 1)y = u_1 - u_2$.

Looking at the proofs of the results in the previous "deterministic" section, however, reveals that this situation is not generic. An arbitrarily small perturbation of u may lead to a situation such that one cannot find a bounded stochastic process y with $P(\sigma)y = Q(\sigma)u$.

CONCLUDING REMARKS

In the book by Hannan and Deistler [1], one can find a different proof of the foregoing result. They work with the Laurent series of $P(s)^{-1}$ converging on a certain annulus containing the unit circle. In a sense our treatment is simpler, but, apart from our totally different proofs, we also have results in the case wherein det $P(\lambda) = 0$ does *not* imply $|\lambda| = 1$. Further, our proofs are totally constructive, although it is not clear at the moment whether our approach starting from the scalar case, including a factorization of polynomials, is computationally less demanding. For instance, the computational complexity of computing the Smith form of a matrix is high. Further, our treatment makes it clear that it is the controllable part of a linear system that determines its bounded u, bounded y stability.

At first sight our results might seem a bit confusing, but a moment's thinking reveal their plausibility: As we do not have a favorite direction of time, causality is not an issue, and one really should expect to find *symmetrical* conditions with respect to the unit circle in the complex plane.

For more details about our polynomial way of studying linear systems the reader is referred to J. C. Willems's papers [3].

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Received 15 March 1989; final manuscript accepted 30 September 1989